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ABSTRACT

Coalitions and vote-trading probabilities exist as ways by which the political process reduces conflict and takes account of intensities of minority preferences. By creating matrices of voters and issues, all possible winning coalition patterns were specified and the probabilities within each case were calculated. A preference vector for each voter was determined by examining the probability of vote-trading within the matrix. It was determined that as the coalition pattern moves from strong dominance by one majority to a multiplicity of majorities, the probability of trading increases. Results can also be used to determine the effect of adding issues on vote-trading possibilities. (CP)

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Coalitions, Minority Representation, and Vote-Trading Probabilities

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COALITIONS, MINORITY REPRESENTATION, AND VOTE-TRADING PROBABILITIES

Edwin T. Haefele*

Although there is disagreement on the normative attributes of vote-trading in legislative bodies (is log-rolling good or bad?), there is little doubt that it exists as one of the ways the political process reduces conflict and takes account of intensities of minority preferences. The existence of coalitions of minorities was posited by Madison (in Federalist Papers no. 10) as fact *and* value. His argument runs along the negative side, i.e., that no tyranny of the majority can exist in the Republic because of the lack of *one* majority on all issues. He neglected (for good reasons) the obverse side of the coin – minorities can band together to pass legislation as well as to defeat legislation. Americans have made good use of vote-trading both to pass and to defeat legislation ever since.

There has been no systematic attempt to relate the possibility of vote-trading to different coalition patterns, however, perhaps because the task is tedious and the theoretical significance (after Madison) was unrecognized until recently.[1] The advent of the computer has reduced the tedium of the task, and the work of Riker [3, Ch. 2] helps to narrow the task considerably. Riker put back into political theory the notion of the minium winning coalition (maximum individual benefit to each member of the winning coalition) and this reduces the number of cases that have theoretical significance.

In brief, Riker's theorem states that "in social situations similar to n-person, zero-sum games with side payments, participants create coalitions just as large as they believe will ensure winning and no larger." [3, pp. 32-33]. We can assume that rational coalition formation will make all coalitions of the minimum winning variety *for the purpose* of comparing vote-trading in different coalition patterns.

Coalition patterns emerge from the bargaining among members of a legislature, committee, or commission on the issues which come before it. The initial coalitions are formed in the bill-drafting stage and determine initial support for each bill. If, for example, we have five legislators and two issues, only three initial patterns are possible under the assumptions of *minimum* winning coalitions and majority rule.

These initial patterns are:

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[1]For a general statement, see Buchanan and Tullock [1]. For the analogy between voting-trading and an economic market, see Coleman [2].

(1) Case 30[2]

Issues \ Voters	1	2	3	4	5	
A	Y	Y	Y	N	N	-- Pass
B	Y	Y	Y	N	N	-- Pass

(2) Case 22

A	Y	Y	Y	N	N	-- Pass
B	Y	Y	N	Y	N	-- Pass

Case 14

A	Y	Y	Y	N	N	-- Pass
B	N	N	Y	Y	Y	-- Pass

[2]Case designations are formed by counting frequency with which voters appear in the initial coalitions, thus:

	Number of voters in 5 coalitions	Number of voters in 4 coalitions	Number of voters in 3 coalitions	Number of voters in 2 coalitions	Number of voters in 1 coalition
Pattern (1)	–	–	–	3	0
Pattern (2)	–	–	–	2	2
Pattern (3)	–	–	–	1	4

This notational scheme was suggested by Elizabeth Duenckel and can be used in designating larger matrices (3 x 5, 4 x 5, 5 x 5) by expanding to the left.

Only seven cases are possible when a third issue is added (3 x 5), eleven cases in the 4 x 5 matrix, and eighteen in the 5 x 5 matrix. Columns permutations can be ignored. Table 1 gives the complete list of cases through the 5 x 5 matrices.

Table 1. Coalition Patterns

Voting Matrix	Case Designation
2 x 5	30, 22, 14
3 x 5	300, 211, 203, 130, 122, 122A,[a] 041
4 x 5[b]	3000, 2101, 2020, 2012, 1210, 1202, 1121, 1040, 0400, 0311, 1230
5 x 5[b]	30000, 21001, 20110, 20102, 20021, 12010, 11200, 12002, 11111, 11030, 10301, 10220, 03100, 03011, 02201, 02120, 01310, 00500

[a]A 122 case with duplication of columns.

[b]Some cases have variants if duplicate columns are allowed.

Having all possible minimum winning coalition patterns specified, the probabilities of trading within each case can be calculated once some means of specifying preferences is decided upon. Since the object of the exercise is to compare trading probabilities among cases, the only requirement of specifying preferences is that the method be consistent across cases and matrices. Four such specifications are used to accomplish this comparison. Each method generates some number of preference vectors for each voter.

A *preference vector* is a column vector composed of 0's, 1's, and -1's which indicates whether or not winning on one issue is more important to the voter than winning on another. Thus a 3 x 5 voting matrix (a case 041),

Voters:	1	2	3	4	5
Issue A	Y	Y	Y	N	N
Issue B	Y	Y	N	Y	N
Issue C	N	N	Y	Y	Y

in which all issues are passing, might have a preference matrix as follows:

-1	0	0	1	0
0	0	1	0	0
1	0	-1	-1	0

where: -1 indicates membership in a winning coalition but a willingness to trade off his vote for another issue on which he is losing;

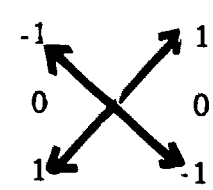
0 indicates (if winning on the issue) an unwillingness to trade it off, or (if losing) an unwillingness to give up any other issue to gain this one;

1 indicates the voter is losing on the issue and is willing to trade another issue for it. One exception to this notation is explained later.

Examining the preference matrix given, we can identify possible trades by first picking out *trading vectors* (preference vectors which have at least one each -1 and 1). Thus, the *trading vectors* are:

Voter 1	Voter 3	Voter 4
-1	0	1
0	1	0
1	-1	-1

and the only trade[3] is between Voters 1 and 4 on issues A and C:



Identification of such trades can be generalized so long as consistent sets of preference vectors are given to each voter.

Preference vectors of the type here being used are the result of combining a given vote vector, e.g., $\begin{matrix} N \\ Y \\ N \end{matrix}$ with an *ordering* or ranking of relative interest in the issues. Such orderings are traditionally given as $\begin{matrix} A \\ B \\ C \end{matrix}$, meaning that of issues A, B,

[3]It should be noted that the trade may or may not be stable. What we are concerned with here is not "solutions" to each "game," but only a test of whether or not any vote-trading possibilities exist.

and C the voter thinks A is most important, B next, and C least important. The *preference vector* which results from $\begin{matrix} N \\ Y \\ N \end{matrix}$ and $\begin{matrix} A \\ B \\ C \end{matrix}$ (assume all issues pass) is $\begin{matrix} 1 \\ -1 \\ 0 \end{matrix}$. If only the order vector was changed, say to $\begin{matrix} C \\ B \\ A \end{matrix}$, the preference vector would be $\begin{matrix} 0 \\ -1 \\ 1 \end{matrix}$

Table 2 sets up four possible ways to generate preference vectors for each voter. The likelihood of any *preference vector* occurring can also be specified, but in the results which follow equal likelihood is assumed unless otherwise noted. (In analyzing real situations, empirical data could be used to make more realistic assumptions about occurrence of certain preference vectors, e.g., one vector twice as likely as another.)

Results

Tabulations of trades "inside the game" and calculation of probabilities is straightforward but tedious if done by hand. A computer program[4] was devised which efficiently both tabulates trades and calculates probabilities. As with most combinatorial problems, however, even computer storage must sooner or later give out, so complete results are limited to the 2 x 5, 3 x 5, and 4 x 5 matrices with a few explorations into the 5 x 5 realm (where the base for probability calculations in the Random Set is 32^5 or 33, 554, 432) and beyond.

The probability of trading is defined as:

$$\frac{\text{number of vector matches}}{\text{number of vector sets}}$$

where a vector set is a selection of one vector from each voter, and the total number of vector sets is V^n where V = number of vectors per voter[5] and n = number of voters. A vector match is defined as any vector set containing at least one trade. The vector matches are counted by an algorithm explained in the Appendix.

[4]Developed by Elizabeth Duenckel, whose perserverance and ingenuity is gratefully acknowledged.

[5]If all voters have an equal number of vectors. If voters' vectors are unequal, then the total number of vector sets is expressed as $\prod_{1}^{n} V$.

Table 2

Preference Vector Sets for 3 x 5 Voting Matrices

	Set	Order Vectors for Generating Preference Vectors
1.	Strong Ordering (6 vectors for each voter)	A A B B C C B C A C A B C B C A B A
2.	Strong Indifference (6 vectors for each voter)	A B C BC CA BA BC CA BA A B C
3.	Rotation (3 vectors for each voter)	A B C B C A C A B
4.	Random (8 vectors for each voter)	Generate all possible preference vectors directly from a given vote vector, e.g., given $\begin{matrix}Y\\Y\\N\end{matrix}$ and all issues winning, the logical combinations* of 0, -1, 1 are: -1 -1 0 0 -1 -1 0 0 -1 0 -1 0 -1 0 -1 0 1 1 1 1 0 0 0 0 *Some of these vectors assume some trading "outside the game", e.g. $\begin{matrix}-1\\0\\0\end{matrix}$ indicates the voter would trade off his vote on issue A, but not for any other issue in the game.

It will be helpful to examine the 3 x 5 matrix first, since the cases are simple, yet show substantial variation typical of larger matrices. Table 3 gives these probabilities and they are plotted on Chart I.

Comparisons among cases within a given preference set (i.e., reading down the columns of Table 3) are the relevant comparisons to make, since the levels of probability of any case *across* preference sets (i.e., the rows of Table 3) are artifacts of the preference sets. It is clear, however, that regardless of which preference set is chosen, as the coalition pattern moves from strong dominance by one majority to a multiplicity of majorities (a la Madison – from case 300 to case 041), the probability of trading increases.

The imputed preference sets can also be used to see what difference adding issues makes to trading probabilities. For example, Chart II shows how trading probabilities increase as the number of issues is increased, using the Random Preference Set.[6]

Although the overall level of probabilities is an artifact of the preference set chosen, some additional evidence of variation as issues are added is given in Chart III which uses the Rotation Set. It should be noted that, no matter what preference set is chosen, the probability of trading can only approach unity. There is always one non-trading vector set – when all preference orderings are identical.

[6]It should be noted that the number of vectors each person can have doubles each time an issue is added, i.e.,

2 issues	=	4	vectors per person,
3 "	=	8	" " " ,
4 "	=	16	" " " ,
5 "	=	32	" " " ,

and that the base for calculating probability goes up as the 5th power of the number of vectors (5 voters). The number of times a particular vector match (potential trade) occurs likewise increases.

Table 3

Probability of Trading
(all calculated on equal likelihood basis
and rounded to 2 decimal places)

3 x 5 Voting Matrix

Case	Strong Ordering Set	Rotation Set	Strong Indifference Set	Random Set
300	zero	zero	zero	zero
211 (2)[a]	.39	.33	.19	.11
203 (3)	.54	.48	.28	.17
130 (3)	.54	.48	.28	.17
122 (4)	.70	.63	.43	.27
122A (4)	.75	.68	.49	.31
041 (5)	.81	.75	.54	.36

[a]Numbers in parentheses indicate number of traders.

Case 300

Y	Y	Y	N	N
Y	Y	Y	N	N
Y	Y	Y	N	N

Case 211

Y	Y	Y	N	N
Y	Y	Y	N	N
Y	Y	N	Y	N

Case 203

Y	Y	Y	N	N
Y	Y	N	Y	N
Y	Y	N	N	Y

Case 130

Y	Y	Y	N	N
Y	Y	N	Y	N
Y	N	Y	Y	N

Case 122

Y	Y	Y	N	N
Y	Y	N	N	Y
Y	N	Y	Y	N

Case 122A

Y	Y	Y	N	N
Y	Y	Y	N	N
Y	N	N	Y	Y

Case 041

Y	Y	Y	N	N
Y	Y	N	Y	N
N	N	Y	Y	Y

CHART I
3 x 5 Matrix

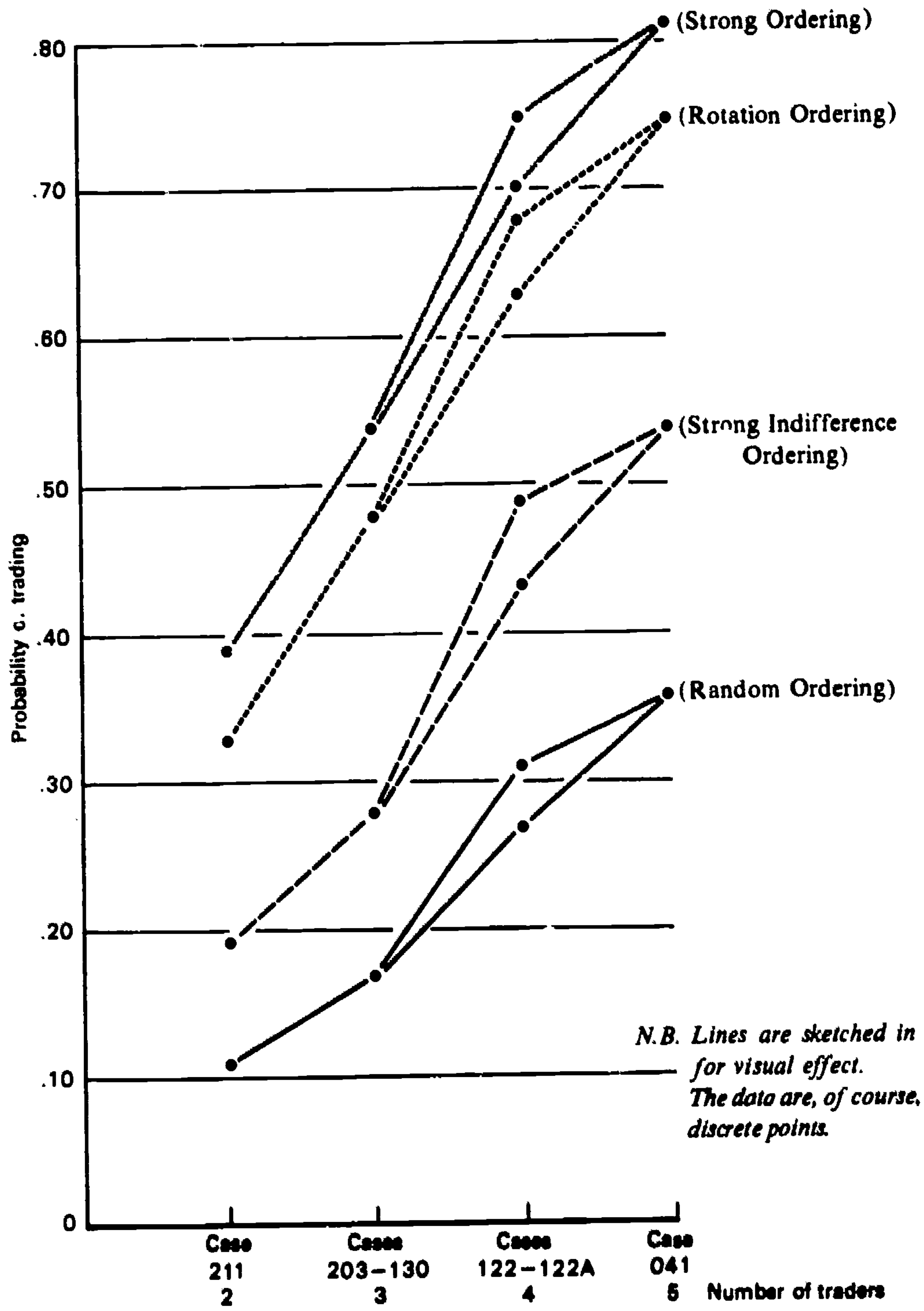
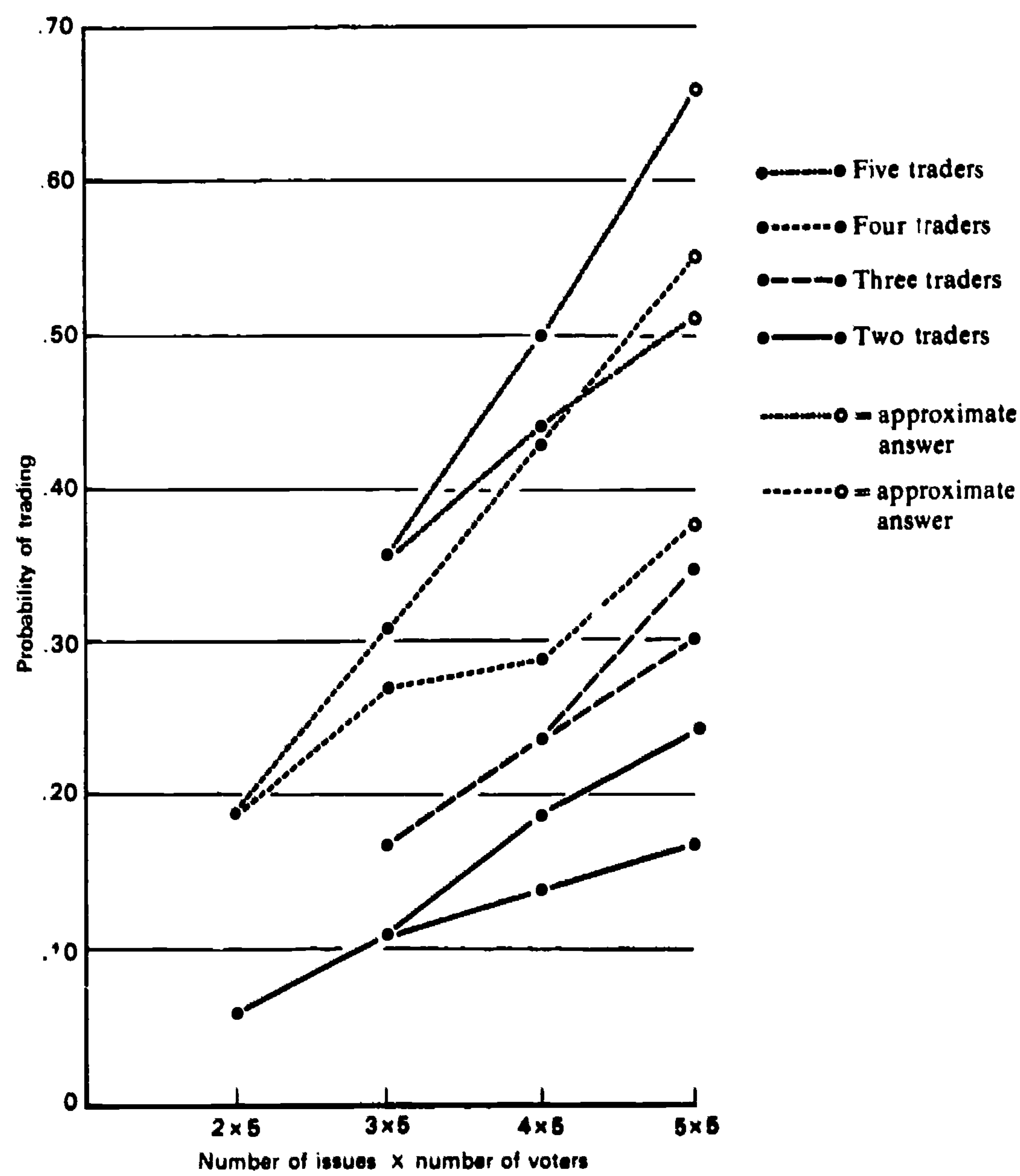
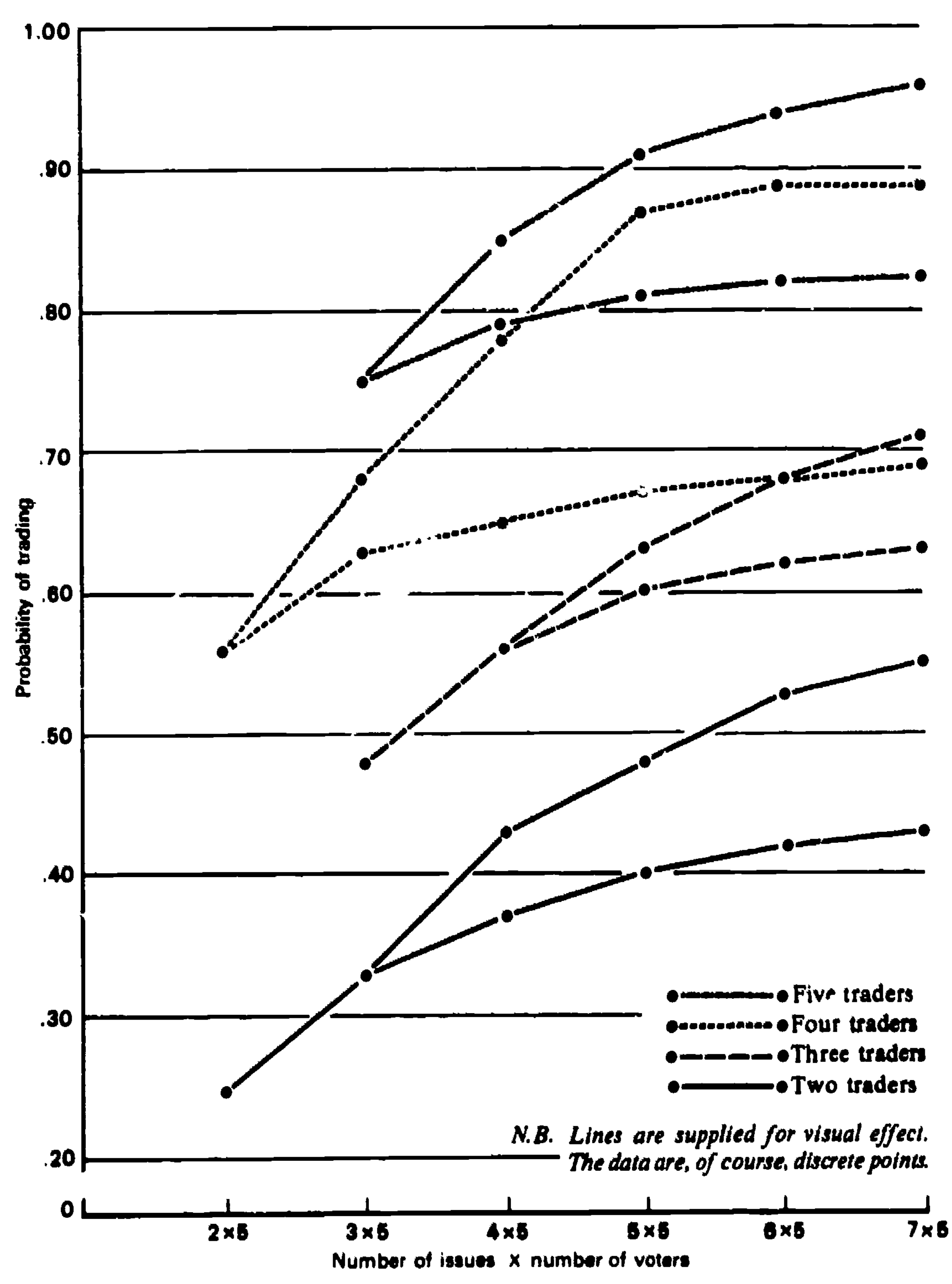


CHART II
Trading Probabilities Related to Issues
(Random Preference Set)



N.B. 1. Lines are supplied for visual effect. The data are, of course, discrete points.
2. Only upper and lower bounds of the range are plotted.

CHART III
Trading Probabilities Related to Issues
(Rotation Preference Set)



Conclusions

In "real" situations, each legislator will have, of course, only one preference vector instead of several, and the probability of trading in any legislature, commission, or committee will be a function of that one vector set. It may be worthwhile, however, when devising new commissions, special districts, or other decision structures to take some note of the number of *independent* issues which are likely to come before such bodies and to examine the probable coalitions and preferences which the members of the body are likely to establish relative to those issues. Thus, the issue of minority representation can be cast in a new light. If a minority representative is not likely to be needed in any minimum winning coalition, his presence does him no good and is frustrating to him. He is essentially powerless as he has nothing to trade. Likewise, if the scope of the decision body is restricted to one issue, so that all matters which come before it are likely to be strongly interdependent, then vote-trading can play only a small role in decision-making. As vote-trading is restricted, the probability of *one* dominant majority rises again with frustrating results for the minority. It also follows, almost without saying, that if the pattern of representation (on the decision body) itself produces *one* dominant majority (i.e., the 300 case), then minority interests are in nowise considered except by the action of altruism, not a reliable defender of minorities.

These considerations may be made clearer by an example. Let us suppose a commission is established to study and make recommendations about water quality in a river. With this its only task, the decisions it takes are essentially mutually exclusive, that is, it faces a set of decisions such that

water quality level A
or water quality level B
or water quality level C

may be chosen.

If there are three municipal and industrial water users and two conservation leaders on this commission, the outcome is fairly clear. Concessions to the conservation leaders would take place, if at all, only because of possible effects outside the commission after the recommendation had been made. This concession would have to be made, regardless, and depends *not* at all on the presence of the conservation interest on the commission. While perhaps self-evident, many boards and commissions function in this fashion, and the equating of "letting minority interests have their say" with democratic process is a commonplace.

If the concern with water quality were placed in a somewhat larger context, let us say an interstate agency to manage a river basin, a different pattern emerges.

With many issues to resolve, it is less likely that one dominant majority on all issues will occur. For example, we could imagine the following agenda and voting matrix:

Bills	State 1 Rep.	State 2 Rep.	State 3 Rep.	State 4 Rep.	State 5 Rep.
Water Quality	Y	Y	N	N	Y
Effluent Charges	Y	Y	N	N	N
Construction of Dams	N	Y	Y	Y	N

Under majority rule, the first and third bills would pass, but the plausible assumption of an order matrix:

A	A	B	A	C
B	C	A	C	B
C	B	C	B	A

yields the preference matrix:

0	0	1	1	-1
0	0	0	-1	-1
0	0	-1	-1	1

in which a trade is possible. State Representative no. 5 can trade either with no. 4 or no. 3. All three issues are hence under pressure, and the potential for striking a bargain all can live with is enhanced.

Governments of general jurisdiction have, of course, the widest selection of independent issues on which trades may be struck. The NEW YORK TIMES headline of 16 July 1969, "Oil Drilling in Alaska? It Could Determine the Senate's Vote on ABM Issue," evokes the range of potential trading at the national level. But even in the governments of general jurisdiction – at local, state, and national levels – we may, as Herbert Gans noted in a recent NEW YORK TIMES article (13 August 1969), have approached a dominant majority problem insofar as race is concerned. If a majority party can be (or has been) constructed without any need for Negro support, the result will leave the Negro powerless at the national level. Even more violence could be the result.

APPENDIX

The Vector Match Counter
by
Elizabeth Duenckel

Identification of variables:

n = number of voters

t = number of traders (voters with at least one trading vector)

P = number of vectors per voter (n x 1)

Q = number of trading vectors per voter (n x 1)

R = number of non-trading vectors per voter (n x 1) ($p_i = q_i + r_i$ $i = 1, \ldots, n$)

V = matrix of all vector sets, each set composed of one vector from each of the n voters

s = total number of vector sets

$$s = \prod_{i=1}^{n} p_i$$

Notation for the vectors of individual voters:

Denote the group of p_i vectors for each voter by ascending letters of the alphabet, i.e., A – set of p_1 vectors for Voter 1

. .
. .
. .

E - - set of p_5 vectors for Voter 5.

1. Partition the p_i vectors for each voter i into two disjoint sets ($i = 1, \ldots, n$), i.e.,

$A_1, \ldots, A_{q_1}$: the trading vectors for Voter 1

$A_{q_1}, \ldots, A_{q_1+r_1}$: the non-trading vectors for Voter 1

2. For $k = 2, \ldots, t$:

   a. Calculate w_k, the number of vector sets of V in which exactly k trading vectors and n-k non-trading vectors occur.

   $$w_k = \prod_{i=1}^{k} q_i \prod_{j=1}^{n-k} r_j$$

   where i designates a voter with a trading vector, j designates a voter with a non-trading vector.

   b. Determine the sets of k trading vectors, one from each of k of the t traders, in which a vector match does not occur.

   c. For each non-match, calculate y, the number of times the non-match occurs in the subset of the w_k vectors of V.

   $$y = \prod_{j=1}^{n-k} r_j$$

   where j is not equal to any of the k traders with a non-matching trading vector. Accumulate, with each calculation of y, z_k, the total number of vector sets with exactly k trading vectors that do not match and n-k non-trading vectors.

   d. Subtract z_k from w_k to get x_k, the number of vector sets with exactly k trading vectors in which at least one vector match occurs.

3. Calculate the total number of vector sets with at least one vector match, m.

$$m = \sum_{j=2}^{t} x_j$$

4. The probability of a vector match is, then, m/s.

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